arXiv:1007.2706v2 [math.GR] 18 Feb 2012

# FINITELY ANNIHILATED GROUPS

MAURICE CHIODO

Abstract. We say a group is *finitely annihilated* if it is the set-theoretic union of all its proper normal finite index subgroups. We investigate this new property, and observe that it is independent of several other well known group properties. For finitely generated groups, we show that in many cases it is equivalent to having non-cyclic abelianisation, and at the same time construct an explicit infinite family of counterexamples to this. We show for finitely presented groups that this property is neither Markov nor co-Markov. In the context of our work we show that the weight of a non-perfect finite group, or a non-perfect finitely generated solvable group, is the same as the weight of its abelianisation. We generalise a theorem of Brodie-Chamberlain-Kappe on finite coverings of groups, and finish with some generalisations and variations of our new definition.

## 1. Introduction

A group is said to be *residually finite* if every non-trivial element lies outside some (proper, normal) finite index subgroup. That is, the intersection of all proper, normal, finite index subgroups is the trivial element. Residually finite groups have been the subject of extensive study. They contain the class of fundamental groups of 3-manifolds, shown by combining a result of Hempel [6] with Perelman's solution to the Geometrization Conjecture [15]. In the case of finitely presented groups, they have solvable word problem [13], and are Hopfian [5]. However, what if we were to reverse this definition somewhat, and instead consider what would happen if we insisted that each element lies *inside* (rather than outside) some proper normal finite index subgroup? We explore this idea in the remainder of this paper.

We say a group is *finitely-annihilated* (abbreviated to F-A) if it is the set-theoretic union of all its proper, normal, finite index subgroups. We use the term F-A because the property is equivalent to following: $G$ is F-A if for every element $g \in G$ there is a finite non-trivial group $H$ and a surjection $\phi : G \twoheadrightarrow H$ with $\phi(g) = e$. That is, each element is annihilated in some non-trivial finite quotient.

It was shown by Brodie-Chamberlain-Kappe in [4, Theorem 1] that a group $G$ is the union of finitely many of its normal subgroups if and only if it has a quotient isomorphic to an elementary $p$-group of rank 2 for some prime $p$;

2010 *AMS Classification:* 20F99.

*Keywords:* coverings of groups, weight of finite groups.

*The author was supported by:*

-An Australian Postgraduate Award.

-A David Hay Postgraduate Writing-Up Award.

$C_p \times C_p$. Using this result, we are able to give the following characterisation of being F-A for some well-known classes of groups:

**Theorem 5.13.** *If $G$ is finitely generated and lies in at least one of the following classes, then $G$ is F-A if and only if $G^{\text{ab}}$ (the abelianisation of $G$) is non-cyclic.*
1. *Free.*
2. *Solvable, and in particular abelian.*
3. *Having only finitely many distinct finite simple quotients, and in particular finite or simple.*
4. *Two generator, with the generators having finite coprime order.*

Note that not all finitely generated groups satisfy the above condition. In [7] Howie was able to show that for distinct primes $p, q, r$, if we define the group $G := \langle x, y, z \mid x^p, y^q, z^r, w \rangle$, where the exponent sums $\exp_x(w)$, $\exp_y(w)$, $\exp_r(w)$ (sums of powers of all instances of $x, y, z$ respectively in $w$) are coprime to $p, q, r$, then $G$ admits a representation $\rho : G \to SO(3)$ with $\rho(x), \rho(y), \rho(z)$ having orders precisely $p, q, r$ respectively. We use this result to construct the following infinite family of groups which lie outside any possible classification like theorem 5.13:

**Theorem 5.15.** *Let $p, q, r$ be distinct primes. Then the group $G := C_p * C_q * C_r$ is F-A, but $G^{\text{ab}} \cong C_{pqr}$ which is cyclic.*

The *weight* of a group $G$, denoted $\mathrm{w}(G)$, is the smallest $n$ such that $G$ is the normal closure of $n$ elements. We are able to use the characterisation of theorem 5.13 to provide an alternative proof to the following, first proved in [11]:

**Corollary 6.2.** *Let $n > 1$, and let $G$ be a finite or solvable group. Then $\mathrm{w}(G) = n$ if and only if $\mathrm{w}(G^{\text{ab}}) = n$; $\mathrm{w}(G) \leq 1$ if and only if $\mathrm{w}(G^{\text{ab}}) \leq 1$.*

We make the observation that being F-A is neither Markov nor co-Markov (corollary 7.5), by showing the following embedding theorem for F-A groups; the question as to whether it is an algorithmically recognisable property amongst finitely presented groups remains open.

**Theorem 7.3.** *Any finitely presented group $G$ which is (resp. is not) F-A can always be embedded into some finitely presented group which is not (resp. is) F-A.*

We can generalise the definition of F-A groups to $n$-coverings (every set of $n$ elements lies in a proper normal finite index subgroup) which we call $n$-F-A. We establish the following generalisation of the result by Brodie-Chamberlain-Kappe in [4, Theorem 1]:

**Theorem 8.8.** *A finitely generated group $G$ has a finite proper $n$-covering $\bigcup_{i=1}^{k} N_i$ by normal finite index subgroups if and only if $\mathrm{w}(G^{\mathrm{ab}}) \geq n+1$ (equivalently, if and only if $G$ surjects onto an elementary $p$-group of rank $n+1$ for some prime $p$).*

With the above result, a lot of our observations about F-A groups generalise to $n$-F-A groups (either trivially, or by expanding the proof in a more general context). Therefore, the above result allows us to extend our charaterisation of F-A groups in theorem 5.13, to get the following characterisation for some $n$-F-A groups.

**Theorem 8.14.** *If $G$ is finitely generated and lies in at least one of the following classes, then $G$ is $n$-F-A if and only if $\mathrm{w}(G^{\mathrm{ab}}) \geq n+1$.*
1. *Free.*
2. *Solvable, and in particular abelian.*
3. *Having only finitely many distinct finite simple quotients, and in particular finite or simple.*

**Acknowledgements**: The author wishes to thank the graduate participants of the conference 'Geometric Group Theory' (Poznan, June 2009) for their initial interest in this question, Rishi Vyas for his many long and thoughtful discussions on the results contained in this paper, and Jack Button for sharing his extensive knowledge of existing results in group theory.

## 2. Preliminaries and definitions

2.1. **Notation.** If $P = \langle X | R \rangle$ is a group presentation with generating set $X$ and relators $R$, then we denote by $\overline{P}$ the group presented by $P$; $P$ is said to be a *finite presentation* if both $X$ and $R$ are finite. If $X$ is a set, then we denote by $X^{-1}$ a set of the same cardinality as $X$ (considered an 'inverse' set to $X$) along with a fixed bijection $\phi : X \to X^{-1}$, where we denote $x^{-1} := \phi(x)$. We write $X^*$ for the set of finite words on $X \cup X^{-1}$, including the empty word $\varnothing$. If $g_1, \ldots, g_n$ are a collection of elements of a group $G$, then we write $\langle g_1, \ldots, g_n \rangle^G$ for the subgroup in $G$ generated by these elements, and $\langle\!\langle g_1, \ldots, g_n \rangle\!\rangle^G$ for the normal closure of these elements in $G$. The *weight* of $G$, $\mathrm{w}(G)$, is the smallest $n$ such that $G = \langle\!\langle g_1, \ldots, g_n \rangle\!\rangle^G$; to remove ambiguity, we set $\mathrm{w}(\{e\}) := 0$. A group $G$ is said to be be *Hopfian* if any surjective homomorphism $f : G \twoheadrightarrow G$ is necessarily injective. If $G$ is a group, then we write $G'$ for the derived subgroup of $G$, and $G^{\mathrm{ab}} := G/G'$ for the abelianisation of $G$, where the commutator $[x, y]$ is taken to be $xyx^{-1}y^{-1}$; a group $G$ is said to be *perfect* if $G = G'$ (equivalently, $G^{\mathrm{ab}} \cong \{e\}$).

2.2. **Definition of finitely annihilated groups.** We now formally define finitely annihilated groups, and hope that the reader will pick up the motivation for this by comparing it with that of a residually finite group as discussed in the introduction.

**Definition 2.1.** Let $G$ be a group. An element $g \in G$ is said to be *finitely annihilated* if there is a finite group $H_g$ and a homomorphism $\phi_g : G \to H_g$ such that $\phi_g(g) = e$ and $\mathrm{Im}(\phi_g) \neq \{e\}$. We say a non-trivial group $G$ is *finitely annihilated* (F-A) if all its non-trivial elements are finitely annihilated. From hereon, we insist that the trivial group is not F-A.

Note that we may drop the requirement that $\mathrm{Im}(\phi_g)$ is non-trivial, and instead insist that $\phi_g$ is a surjection to a non-trivial finite group $H$; this is clearly equivalent. We often find the following definition, easily shown to be equivalent to the above, to be much more useful in the study of such groups.

**Definition 2.2.** A group $G$ is F-A if and only if for each $g \in G$ there exists a proper, normal subgroup $N$ of finite index in $G$ such that $g \in N$. That is, $G$ is the union of all its proper, normal, finite index subgroups.

The following is then immediate, which we state without proof.

**Proposition 2.3.** *Definitions* 2.1 *and* 2.2 *are equivalent.*

We say that a normal subgroup $N \lhd G$ is *maximal normal* if, whenever $M \lhd G$ and $N \subseteq M$, then either $M = N$ or $M = G$. Equivalently, $G/N$ is simple.

**Proposition 2.4.** *A group $G$ is F-A if and only if it is the union of all its maximal normal, proper, finite index subgroups.*

*Proof.* Suppose $G$ is F-A. Let $N \lhd G$ be proper and of finite index. Then the finite group $G/N$ is either simple (in which case $N$ is maximal normal in $G$), or has a maximal normal, proper subgroup whose preimage in $G$ is maximal normal, proper, and contains $N$. So we can replace each such $N$ by a maximal normal, proper, finite index subgroup containing it. The converse is immediate. □

From hereon we will usually find it convenient to use the covering by all maximal normal, proper, finite index subgroups when working with F-A groups.

## 3. Elementary observations

We now note some necessary and sufficient conditions for a group to be F-A.

**Proposition 3.1.** *Let $G$ be a non-trivial group. Then $G$ is F-A if and only if neither of the following hold:*
1. *$G$ has weight* 1.
2. *There is some $g \in G$ such that $G/\langle\langle g \rangle\rangle^G$ has no proper finite index subgroups.*

*Proof.* Suppose $G$ is F-A. Then, for each $g \in G$, $G/\langle\langle g \rangle\rangle^G$ must have a non-trivial finite quotient, thus neither condition can hold. Conversely, if neither of the two conditions hold, then for any $g \in G$ we must have that $G/\langle\langle g \rangle\rangle^G$ is non-trivial and has a finite quotient, so $G$ is F-A. □

**Corollary 3.2.** *Let $G$ be a finitely generated group with no infinite simple quotients. Then $G$ is F-A if and only if* $\mathrm{w}(G) > 1$.

**Proposition 3.3.** *Let $G$ be a finitely generated group which is neither F-A nor weight* 1. *Then $G$ has an infinite simple quotient.*

*Proof.* If $\mathrm{w}(G) > 1$, then by proposition 3.1 there exists $g \in G$ with $G/\langle\langle g\rangle\rangle^G$ having no proper finite index subgroups. So either this is simple, or itself has a proper normal subgroup $H_1$ of infinite index. Then this quotient by $H_1$ is simple, or has a proper normal subgroup $H_2$ of infinite index. Continuing in this manner we get $H_1, H_2, \ldots$. Each $H_i$ has a preimage in $G$, call this $\tilde{H}_i$, all normal in $G$. We note that $\langle\langle g\rangle\rangle^G \lhd \tilde{H}_1 \lhd \tilde{H}_2 \lhd \ldots$. But $G$ is finitely generated, so the normal subgroup $H = \bigcup_{i\in\mathbb{N}} \tilde{H}_i$ is necessarily of infinite index, and moreover $G/H$ is simple. □

We now observe that the property of being F-A is neither implied by, nor implies, several well-known algebraic properties of groups. It is also a property not preserved by HNN extensions or quasi-isometries.

**Definition 3.4.** Let $G$ act on a space $X$. The action is said to be *faithful* if, for any two distinct $g, h \in G$, there exists $x \in X$ such that $gx \neq hx$ (equivalently, for any $g \neq e$ in $G$ there exists $x \in X$ such that $gx \neq x$). The action is said to be *free* if, for any two distinct $g, h \in G$ and all $x \in X$, we have $gx \neq hx$ (equivalently, if $gx = x$ for some $x \in X$ then $g = e$).

**Proposition 3.5.** *Let $G$ act faithfully by automorphisms on a group $H$. Then $G$ being F-A is, in general, independent of $H$ being F-A.*

*Proof.* $\mathbb{Z}_2$ is not F-A, but acts faithfully (by inversion) on any group. Conversely, take the action of $C_2 \times C_2 = \langle a, b \mid a^2 = e, b^2 = e, ab = ba\rangle$ on $C_3 \times C_5 = C_{15} = \langle c, d \mid c^3 = e, d^5 = e, cd = dc\rangle$, induced via $a \cdot (c^n, d^m) = (c^{-n}, d^m)$ and $b \cdot (c^n, d^m) = (c^n, d^{-m})$. Then this is a faithful action by automorphisms. But $C_2 \times C_2$ is F-A, whereas $C_{15}$ is cyclic, thus not F-A by proposition 3.1. □

**Proposition 3.6.** *Let $G$ act freely and faithfully on a group $H$. Then $G$ being F-A is, in general, independent of $H$ being F-A.*

*Proof.* We first observe that if $H \leq G$, then $H$ acts on $G$ freely and faithfully, by left multiplication. So now we merely note that $C_2$ (not F-A) is a subgroup of $C_2 \times C_2$ (which is F-A), and $C_2 \times C_2$ (which is F-A) is a subgroup of $S_5$ (which is not F-A by proposition 3.1, since it is simple and hence of weight 1). □

As we see below, being F-A is rather independent of many other group properties (we leave the proof of the following to the reader).

**Proposition 3.7.** *There is no implication (in either direction) between being F-A and having any of the following group properties:*
1. *Being finite*
2. *Being residually finite*
3. *Having solvable word problem*

The author is grateful to Benno Kuckuck for pointing out the second part of the proof of the following proposition.

**Proposition 3.8.** *Being F-A is neither a quasi-isometry invariant, nor preserved by HNN extensions.*

*Proof.* Clearly $C_2$ is not F-A, however $C_2 \times C_2$ is, and they are both finite (thus quasi-isometric). For the second part, let $\langle a, b \mid [a,b] = e \rangle$ be a presentation for $\mathbb{Z}^2$. Let $A = \langle a^2, b^2 \rangle$ and let $B = \langle a^3, b^3 \rangle$. It is clear that $A \cong B$, as they are both isomorphic to $\mathbb{Z}^2$. Form the HNN extension $\overline{K}$ of $\mathbb{Z}^2$ with stable letter $t$ by identifying $A$ and $B$, given by the following presentation:

$$K := \langle a, b, t \mid [a,b] = e, t^{-1}a^2t = a^3, t^{-1}b^2t = b^3 \rangle$$

Then $\overline{K}$ is not F-A. This can be most easily seen by observing that $t \neq e$ in $\overline{K}$, but $\langle\langle t \rangle\rangle^{\overline{K}} = \overline{K}$ (as $\overline{K}/\langle\langle t \rangle\rangle^{\overline{K}} = \overline{\langle a, b \mid [a,b] = e, a^2 = a^3, b^2 = b^3 \rangle} \cong \{e\}$), hence by proposition 3.1 $\overline{K}$ is not F-A. □

## 4. Stronger observations

We now move on to develop some important tools and results regarding F-A groups, as well as give important examples. These will be very useful in later sections when we attempt to characterise finitely generated F-A groups.

4.1. **Quotients.** Looking at quotients is an important tool in understanding F-A groups. Here we give some connections between being F-A and taking quotients.

**Theorem 4.1.** *Let $G$ be a group for which there is some quotient $G/H$ which is F-A. Then $G$ itself is F-A.*

*Proof.* $G/H$ is F-A, thus can be written as $G/H = \bigcup_{i \in I} N_i$, where each $N_i$ is proper, normal and of finite index. Let $\phi : G \twoheadrightarrow G/H$ be the quotient map. Then $G = \phi^{-1}(\bigcup_{i \in I} N_i) = \bigcup_{i \in I} \phi^{-1}(N_i)$. Moreover, each $\phi^{-1}(N_i)$ is proper, normal, and of finite index in $G$, as $\phi$ is a surjection. Thus $G$ is F-A. □

That is, being F-A is preserved under reverse quotients.

**Corollary 4.2.** *Let $A$ be a F-A group, and $G$ any arbitrary group. Then $A * G$ and $A \times G$ will also be F-A.*

*Proof.* This is immediate from theorem 4.1 since in each case there is a projection onto $A$, which is F-A. □

The following gives a useful set of sufficient conditions which ensure that being finitely annihilated is preserved under quotients.

**Theorem 4.3.** *Let $G$ be a finitely generated F-A group, and $N \triangleleft G$. If $G = \bigcup_{i \in I} N_i$ is a covering by proper, normal, finite index subgroups, and $N$ is contained in every $N_i$, then $G/N$ is F-A.*

*Proof.* Take the quotient map $f : G \twoheadrightarrow G/N$. Then $f(N_i) = N_i/N$ will be normal and of finite index in $G/N$, as $f$ is a surjection. But since $N \triangleleft N_i$ by hypothesis, we have that $(G/N)/(N_i/N) \cong G/N_i$, hence $f(N_i)$ is also proper in $G/N$. So we have

$$G/N = f(G) = f\left(\bigcup_{i \in I} N_i\right) = \bigcup_{i \in I} f(N_i)$$

and hence $\bigcup_{i \in I} f(N_i)$ is our desired proper, normal, finite index covering of $G/N$. □

We will make very frequent use of the above two theorems later on, when finding an alternate characterisation of F-A groups.

4.2. **Examples.**

**Proposition 4.4.** *No non-trivial simple group $S$ is F-A.*

*Proof.* Any non-trivial normal subgroup of $S$ is all of $S$, and thus not a proper subgroup. ☐

**Proposition 4.5.** *Let $X$ be a set. Then the free group on $X$, $F_X$, is F-A if and only if $|X| \geq 2$.*

*Proof.* If $|X| \geq 2$ then $F_X$ clearly surjects onto $F_2$, which in turn surjects onto $C_2 \times C_2$, a group which is F-A. Thus $F_X$ is F-A by theorem 4.1. Conversely, if $|X| \leq 1$, then $F_X$ is cyclic, and hence not F-A by proposition 3.1. ☐

**Proposition 4.6.** *A free product of a group $S$ having no proper normal finite index subgroups (in particular an infinite simple group), and a group $G$ of weight $1$, is never F-A.*

*Proof.* Since $\mathrm{w}(G) = 1$, there exists $g \in G$ such that $\langle\!\langle g \rangle\!\rangle^G = G$. Hence $\langle\!\langle g \rangle\!\rangle^{G*S} = \langle\!\langle G \rangle\!\rangle^{G*S}$, and so $(G * S)/\langle\!\langle g \rangle\!\rangle^{G*S} = (G * S)/\langle\!\langle G \rangle\!\rangle^{G*S} \cong S$. Suppose $N$ is a proper, normal, finite index subgroup of $G * S$ containing $g$. Then $N$ contains $\langle\!\langle G \rangle\!\rangle^{G*S}$, and so

$$(G * S)/N \cong \left((G * S)/\langle\!\langle G \rangle\!\rangle^{G*S}\right) / \left(N/\langle\!\langle G \rangle\!\rangle^{G*S}\right) \cong S/\left(N/\langle\!\langle G \rangle\!\rangle^{G*S}\right)$$

So $S$ has a proper, normal, finite index subgroup, which is impossible. ☐

Up to now, we have usually shown that a group is not F-A by showing that it is the normal closure of one element. We now give examples of groups which are neither F-A nor the normal closure of one element. Of course, $\mathbb{Q}$ would be an obvious example, as it has no finite index subgroups. However, we construct a finitely generated example. To do this we require a few preliminary results. The first of these is by Ol'Shanskii, found as Theorem 28.3 in [14].

**Theorem 4.7** (Ol'Shanskii [14, Theorem 28.3])**.** *There exists a $2$-generator infinite simple group $S$ in which every proper subgroup is infinite cyclic (and hence $S$ is torsion-free).*

The next is a (deep) partial result on the Kervaire conjecture, found as Theorem A in [10].

**Theorem 4.8** (Klyachko [10, Theorem A])**.** *Let $G$ be torsion-free and non-trivial. Then the group $G * \mathbb{Z}$ has weight at least $2$.*

The following result, showing that being F-A is not equivalent to having weight greater than 1, was inspired by a correspondence between the author and Mathieu Carette.

**Proposition 4.9.** *There is a $3$-generator group which is neither F-A nor the normal closure of any one element.*

*Proof.* Take the infinite, 2-generator, torsion-free, simple group $S$ from theorem 4.7, and form $H := S * \mathbb{Z}$. Then $H$ is neither the normal closure of any single element (theorem 4.8), nor F-A (proposition 4.6). ☐

## 5. Identification and classification

In this section we describe a straightforward method to determine if a group is F-A, provided it lies in some particular collection of classes of groups.

5.1. **Key theorem.** The following result (Theorem 1 in [4]) suggests that there is a strong relationship between being F-A and having non-cyclic abelianisation.

**Theorem 5.1** (Brodie-Chamberlain-Kappe [4, Theorem 1])**.** *A group $G$ has a non-trivial finite covering by normal subgroups if and only if it has a quotient isomorphic to an elementary $p$-group of rank 2 for some prime $p$.*

**Corollary 5.2.** *Let $G$ be a group that can be expressed as the union of finitely many proper, normal, finite index subgroups (and thus is F-A). Then $G$ has a quotient isomorphic to $C_p \times C_p$ for some prime $p$.*

Using the above result, we can easily characterise all the finite F-A groups. We will eventually turn our attention to the following question: Can a finitely generated group have a cover by infinitely many proper, normal, finite index subgroups, but no cover by finitely many such subgroups? That is, are finitely generated groups 'compact' in the sense of coverings in this way. Later we will see that such groups do in fact exist.

5.2. **Finite characterisation.**

**Proposition 5.3.** *Let $G$ be a finitely generated group with only finitely many distinct finite simple quotients. Then $G$ is F-A if and only if $G^{\mathrm{ab}}$ is non-cyclic.*

*Proof.* If $G^{ab}$ is non-cyclic then it surjects onto $C_p \times C_p$ for some prime $p$, and hence is F-A by lemma 5.6 (note this is not create a circular argument, as we do not use this proposition again before proving lemma 5.6). Conversely, if $G$ is F-A, then it can be written as the union of all its maximal normal, proper, finite index subgroups (proposition 2.4). But as $G$ is finitely generated, it can only have finitely many subgroups of a given index. Since the index of these maximal normal, proper, finite index subgroups is bounded, there can only be finitely many of them. So by corollary 5.2, $G \twoheadrightarrow C_p \times C_p$ for some prime $p$, and hence $G^{\mathrm{ab}}$ is non-cyclic. □

**Corollary 5.4.** *A finite group $G$ is F-A if and only if $G^{\mathrm{ab}}$ is non-cyclic.*

5.3. **Abelian characterisation.**

**Lemma 5.5.** *Let $G$ be a finitely generated abelian group. Then $G$ is non-cyclic if and only if it surjects onto $C_p \times C_p$ for some prime $p$.*

*Proof.* Suppose $G$ is cyclic. Then any quotient of $G$ will be cyclic, and thus cannot be $C_p \times C_p$. Now suppose $G$ is non-cyclic. Then by the structure theorem for finitely generated abelian groups (see [8, §I Theorem 2.1]), $G \cong \mathbb{Z}^r \times C_{d_1} \times \ldots \times C_{d_n}$, where $d_i$ divides $d_{i+1}$ for $1 \leq i \leq n-1$. Since $G$ is non-cyclic, then $r + n \geq 2$. In the case $r \geq 2$, we have $G \twoheadrightarrow \mathbb{Z}^2 \twoheadrightarrow C_2 \times C_2$. In the case $r = 1$ take $p$ to be any prime factor of $d_1$. Then $G \twoheadrightarrow \mathbb{Z} \times C_{d_1} \twoheadrightarrow C_p \times C_p$. In the final case $r = 0$, take $p$ to be any prime factor of $d_1$ (and thus a factor of $d_2$). Then $G \twoheadrightarrow C_{d_1} \times C_{d_2} \twoheadrightarrow C_p \times C_p$. □

Observe that if we relax the finitely generated condition, then the above lemma is clearly false. Take $\mathbb{Q}$ for example; this is clearly non-cyclic, but does not surject to $C_p \times C_p$ for any $p$ as it does not have any proper finite index subgroups.

**Lemma 5.6.** *$C_p \times C_p$ is F-A for any prime $p$.*

*Proof.* Given any $g \in G := C_p \times C_p$, we take $\langle g \rangle^G$, which is normal (since $G$ is abelian), proper (since $G$ is non-cyclic), and of finite index (since $G$ is finite). So writing $G = \bigcup_{g \in G} \langle g \rangle$ gives a covering by proper, normal, finite index subgroups; thus G is F-A. □

We make very frequent use of the following corollary, which follows immediately from the above lemma and theorem 4.1.

**Corollary 5.7.** *Suppose a finitely generated group $G$ surjects onto $C_p \times C_p$ for some prime $p$ (equivalently, $G^{\mathrm{ab}}$ is non-cyclic). Then $G$ is F-A.*

We can now provide a complete characterisation of finitely generated abelian F-A groups. Again, relaxing the finitely generated condition gives us a counterexample, namely $\mathbb{Q}$.

**Proposition 5.8.** *Let $G$ be a finitely generated abelian group. Then $G$ is F-A if and only if it is non-cyclic. That is, $G$ is F-A if and only if $G$ surjects onto $C_p \times C_p$ for some prime $p$.*

*Proof.* If $G$ is cyclic then it is not F-A by theorem 3.1. Conversely, suppose $G$ is non-cyclic. Then by lemma 5.5, $G$ surjects onto $C_p \times C_p$ for some prime $p$, and so is F-A by corollary 5.7. □

5.4. **Solvable characterisation.** We can extend our characterisation of finitely generated abelian F-A groups to the class of solvable groups.

**Proposition 5.9.** *Let $G$ be a non-trivial finitely generated group whose finite simple quotients are all abelian (hence finite cyclic). Then $G$ is F-A if and only if $G^{\mathrm{ab}}$ is non-cyclic.*

*Proof.* If every finite simple quotient of $G$ is abelian, then $G'$ is a subgroup of every maximal normal finite index subgroup of $G$. So if $G$ is F-A, then by theorem 4.3, $G^{\mathrm{ab}}$ is F-A, and hence non-cyclic by proposition 5.8. Conversely, if $G^{\mathrm{ab}}$ is non-cyclic, then by proposition 5.8, $G$ is F-A. □

**Corollary 5.10.** *Let $G$ be a non-trivial finitely generated solvable group. Then $G$ is F-A if and only if $G^{\mathrm{ab}}$ is non-cyclic.*

*Proof.* Any finite simple quotient of a solvable group will be abelian, so just apply proposition 5.9. □

5.5. **Characterisation of some other groups.** The following lemma was observed in conjunction with Tharatorn Supasiti.

**Lemma 5.11.** *Let $m, n$ be coprime positive integers. Then $\mathrm{w}(C_m * C_n) = 1$, and hence $C_m * C_n$ is not F-A.*

*Proof.* Let $P = \langle a, b \mid a^m, b^n \rangle$ be a finite presentation for $C_m * C_n$. Then $\overline{P}/\langle\langle ab^{-1} \rangle\rangle^{\overline{P}} \cong \overline{\langle a, b \mid a^m, b^n, ab^{-1} \rangle}$. Since $\gcd(m, n) = 1$, there exist $j, k \in \mathbb{N}$ such that $jm + kn = 1$. So $b^1 = b^{jm+kn} = b^{jm}b^{kn} = e$ in $\overline{P}/\langle\langle ab^{-1} \rangle\rangle^{\overline{P}}$, hence $a = e$, and so the quotient is trivial. □

**Corollary 5.12.** *Let $G$ be a two generator group, where the generators are torsion and of coprime order. Then $G$ is not F-A.*

*Proof.* Any such group $G$ with generators of coprime orders $m, n$ is a quotient of $C_m * C_n$. So if such a group was F-A, then $C_m * C_n$ would be F-A by theorem 4.1, contradicting lemma 5.11. □

5.6. **A characterisation summary.** We summarise our characterisation results so far:

**Theorem 5.13.** *If $G$ is finitely generated and lies in at least one of the following classes, then $G$ is F-A if and only if $G^{\mathrm{ab}}$ (the abelianisation of $G$) is non-cyclic.*
1. *Simple (proposition 4.4)*
2. *Free (proposition 4.5)*
3. *Abelian (proposition 5.8)*
4. *Solvable (corollary 5.10)*
5. *Having only finitely many distinct finite simple quotients (proposition 5.3)*
6. *Finite (corollary 5.4)*
7. *Two generator, with the generators having finite coprime order (corollary 5.12)*

It would, at this stage, be very tempting to naively pose the following two equivalent conjectures.

**Naive Conjecture 1.** Let $G$ be a finitely generated group. Then $G$ is F-A if and only if $G^{\mathrm{ab}}$ is non-cyclic.

**Naive Conjecture 2.** Let $G$ be a finitely generated group. Then $G$ can be expressed as the union of all its proper, normal, finite index subgroups if and only if a finite subcover of these subgroups also covers $G$.

*Proof of the equivalence of conjectures 1 and 2.* Assume conjecture 1 holds. Given a finitely generated group $G$ which can be expressed as the union of all its proper, normal, finite index subgroups, we thus have that $G$ is F-A. So by conjecture 1, $G^{\mathrm{ab}}$ is non-cyclic, so by lemma 5.5, $G$ surjects onto $C_p \times C_p$ for some prime $p$ (say via the map $f : G \twoheadrightarrow C_p \times C_p$). Take a finite covering $C_p \times C_p = \bigcup_{i=1}^n N_i$ by proper, normal, finite index subgroups. Then $G = f^{-1}(C_p \times C_p) = f^{-1}(\bigcup_{i=1}^n N_i) = \bigcup_{i=1}^n f^{-1}(N_i)$ is a covering by proper, normal, finite index subgroups. The reverse direction is immediate.
Now assume conjecture 2 holds. Let $G$ be a finitely generated group. If $G^{\mathrm{ab}}$ is non-cyclic then by corollary 5.7 we have that $G$ is F-A. Conversely, if $G$ is F-A, then by conjecture 2, $G$ has a finite covering by proper, normal, finite index subgroups. So by corollary 5.5 $G$ surjects onto $C_p \times C_p$ for some prime $p$, and hence $G^{\mathrm{ab}}$ is non-cyclic. □

Of course, these conjectures fail immediately if we drop the requirement of being finitely generated (i.e., $\mathbb{Q}$, or $A_5 \oplus A_5 \oplus \ldots$). We are very interested in the classes of finitely generated groups which satisfy these equivalent conjectures. Note that not all do, as the following theorem from [7] shows (pointed out to the author by Jack Button).

**Theorem 5.14** (Howie, [7, Theorem 4.1])**.** *Let $w \in \{x,y,z\}^*$, and define $P := \langle x,y,z \mid x^p, y^q, z^r, w\rangle$ where $p,q,r$ are distinct primes, and the exponent sums $\exp_x(w)$, $\exp_y(w)$, $\exp_r(w)$ (sums of powers of all instances of $x,y,z$ respectively in $w$) are coprime to $p,q,r$ respectively. Then there exists a representation $\rho : \overline{P} \to SO(3)$ with $\rho(x), \rho(y), \rho(z)$ having orders precisely $p,q,r$ respectively.*

Using this result, we can construct the following infinite family of counterexamples to conjecture 1.

**Theorem 5.15.** *Let $p,q,r$ be distinct primes. Then the group $\overline{K} \cong C_p * C_q * C_r$ with presentation $K := \langle x,y,z \mid x^p, y^q, z^r\rangle$ is F-A, but $\overline{K}^{\mathrm{ab}} \cong C_{pqr}$ which is cyclic.*

*Proof.* This closely follows the proof of [7, Corollary 4.2]. Take a word $w \in \{x,y,z\}^*$ and define $P := \langle x,y,z \mid x^p, y^q, z^r, w\rangle$, hence $\overline{P} \cong \overline{K}/\langle\langle w\rangle\rangle^{\overline{K}}$ with associated quotient map $h : \overline{K} \twoheadrightarrow \overline{P}$. If $p$ divides $\exp_x(w)$, then $\overline{P}/\langle\langle y,z\rangle\rangle^{\overline{P}} \cong C_p$, and $w$ is trivial in this (finite) quotient. A similar argument works for when $q$ divides $\exp_y(w)$ or $r$ divides $\exp_z(w)$. Thus $w$ is a finitely annihilated element of $\overline{K}$ in any of these 3 cases. So we are left with the case where $\exp_x(w), \exp_y(w), \exp_r(w)$ are each coprime to $p,q,r$ respectively. Now we may apply theorem 5.14 to show that there is a representation $\rho : \overline{P} \to SO(3)$ which preserves the orders of $x,y,z$ (ensuring they are non-trivial in the image). But then the (non-trivial) image of $\rho$ in $SO(3)$ will be residually finite (as it is a discrete subgroup of a linear group). So, since $|\rho(x)| = p > 1$, then there is a finite group $H$ and a map $f : \mathrm{Im}(\rho) \to H$ with $f(\rho(x)) \neq e$. So the map $f \circ \rho \circ h : \overline{K} \to H$ annihilates $w$, and is a non-trivial homomorphism to a finite group. Thus $w$ is a finitely annihilated element in this last case, so $\overline{K}$ is F-A. $\square$

With this in mind, we make the following definition.

**Definition 5.16.** We say a finitely generated group $G$ is *easily finitely annihilated* if $G^{\mathrm{ab}} \twoheadrightarrow C_p \times C_p$ for some prime $p$. We say $G$ is *just finitely annihilated* if $G$ is F-A, but no proper quotient of $G$ is F-A.

So far, very few of the F-A groups we have come across are just F-A (as we have shown that most have a surjection to $C_p \times C_p$ for some prime $p$). However, theorem 5.15 gives an example of a finitely presented group which is not easily F-A. We would like to understand the class of just F-A groups, as they seem to be the most interesting ones. This removes the obvious examples where we take an arbitrary group $G$ and form $G \times C_2 \times C_2$ to make a F-A group.

**Question 1.** Does there exist a finitely generated, perfect, F-A group?

We suspect the answer to the above question to be no. At this point it makes sense to mention a closely related open problem in group theory, first posed by J. Wiegold as Question 5.52 in [9]:

**Question** (Wiegold [9, Question 5.52])**.** Is every finitely generated perfect group necessarily weight 1?

If the answer to this is yes, then the answer to question 1 would be no (by proposition 3.1); we show the answer to this is yes in the case of finite groups, in corollary 6.3.

## 6. Applications to group theory

We now apply some of our results to prove various facts about groups.

**Theorem 6.1.** *Let $n > 1$, and $G$ be a finitely generated group from a class given in theorem 5.13 such that $G$ has no infinite simple quotients. Then $\mathrm{w}(G) = n$ if and only if $\mathrm{w}(G^{\mathrm{ab}}) = n$; $\mathrm{w}(G) \leq 1$ if and only if $\mathrm{w}(G^{\mathrm{ab}}) \leq 1$.*

*Proof.* We always have $\mathrm{w}(G^{\mathrm{ab}}) \leq \mathrm{w}(G)$ since $G^{\mathrm{ab}} = G/G'$ is a quotient of $G$. Now consider the case where $G$ is in a class that is preserved under taking quotients (i.e., not class 2 or 7). If $\mathrm{w}(G^{\mathrm{ab}}) \leq 1$ then, since $G$ belongs to a class from theorem 5.13, we have that $G$ is not F-A. But $G$ has no infinite simple quotients, so corollary 3.2 shows that $\mathrm{w}(G) \leq 1$. If on the other hand $\mathrm{w}(G^{\mathrm{ab}}) = n > 1$, then take $n$ elements $g_1G', \dots, g_nG'$ whose normal closure is all of $G/G'$. Setting $K := G/\langle\!\langle g_1, \dots, g_{n-1} \rangle\!\rangle^G$ we see that $\mathrm{w}(K^{\mathrm{ab}}) = 1$, and hence $\mathrm{w}(K) = 1$ by what we have just shown. So $\mathrm{w}(G) \leq (n-1) + 1 = n$. But $\mathrm{w}(G^{\mathrm{ab}}) = n$, so $\mathrm{w}(G) \geq n$. Combining these gives that $\mathrm{w}(G) = n = \mathrm{w}(G^{\mathrm{ab}})$.
Finally, for the case where $G$ is in class 2 (free) or class 7 (two-generator, with the generators having finite coprime order), the inequality follows from elementary group theory. $\square$

Seeing as the class of finite groups is listed in theorem 5.13 we have the following immediate corollary, which is another way to resolve the Wiegold question for finite groups (already known in the literature, as a consequence of the main result in [11]).

**Corollary 6.2.** *Let $n > 1$, and let $G$ be a finite or solvable group. Then $\mathrm{w}(G) = n$ if and only if $\mathrm{w}(G^{\mathrm{ab}}) = n$; $\mathrm{w}(G) \leq 1$ if and only if $\mathrm{w}(G^{\mathrm{ab}}) \leq 1$.*

**Corollary 6.3.** *Let $G$ be a non-trivial finite group. If $G$ is perfect (i.e., $\mathrm{w}(G^{\mathrm{ab}}) = 0$), then $\mathrm{w}(G) = 1$. That is, $G$ is the normal closure of one element. So the Wiegold question is resolved for the case of finite groups.*

We note that what we have done here is very similar to the following result by Kutzko in [11], but with a different method of proof.

**Theorem 6.4** (Kutzko [11])**.** *Let $G$ be a group of finite weight and let $L$ be the lattice of normal subgroups of $G$ which are contained in $G'$. Then if $L$ satisfies the minimum condition, $\mathrm{w}(G) = \mathrm{w}(G^{\mathrm{ab}})$ (where Kutzko defines $\mathrm{w}(\{e\}) := 1$).*

## 7. Algorithmic recognisability

We investigate the question of whether finitely presented F-A groups are algorithmically recognisable. The following two results, originally due to Higman, can be found in [16].

**Theorem 7.1** (Higman [16, p. 9])**.** *Define the Higman group $H$ by*

$$H := \left\langle a, b, c, d \mid aba^{-1} = b^2, bcb^{-1} = c^2, cdc^{-1} = d^2, dad^{-1} = a^2 \right\rangle$$

*Let $\phi : H \to G$ be a homomorphism to some group $G$. Then $\phi(a), \phi(b), \phi(c), \phi(d)$ all have finite order if and only if they are all trivial.*

**Corollary 7.2.** *The Higman group $H$ has no proper subgroup of finite index.*

*Proof.* Let $G \leq H$ be a finite index subgroup. Then $G$ has a subgroup $K$ which is normal in $H$ and of finite index. So we have the projection map $\phi_K : H \to H/K$, where $H/K$ is finite. The images of $a, b, c, d$ under this map all have finite order, and thus are trivial by theorem 7.1. But $H/K$ is generated by these images, and is thus trivial. So $K = H$, and hence $G = H$. □

The author wishes to thank Rishi Vyas for his contribution to the second part of the proof of the following theorem.

**Theorem 7.3.** *Any finitely presented group $G$ which is (resp. is not) F-A can always be embedded into some finitely presented group which is not (resp. is) F-A.*

*Proof.* Let $G$ be a finitely presented group which is not F-A. Then $G$ embeds into $G \times C_2 \times C_2$, which by corollary 4.2 is F-A since $C_2 \times C_2$ is F-A. Conversely, let $G = \langle X | R \rangle$ be any finitely presented group. We show that $G$ embeds into some finitely presented group which is not F-A. Take the free product of $G$ with the Higman group $H$, which has presentation

$$G * H = \left\langle X, a, b, c, d \mid R, aba^{-1} = b^2, bcb^{-1} = c^2, cdc^{-1} = d^2, dad^{-1} = a^2 \right\rangle$$

Now form the Adian-Rabin group $(G * H)(a)$ over the word $a$ (see [13, Lemma 3.6]). This group has no proper, finite index subgroups. For suppose so, then it would have a proper, normal, finite index subgroup $K$. Then $(G * H)(a)/K$ is finite, so by theorem 7.1, the image of $a$ in this quotient is trivial. But by the Adian-Rabin relations, this means that the entire quotient is trivial. Hence $K = (G * H)(a)$. Finally, since $a \neq e$, we observe that $G \hookrightarrow G * H \hookrightarrow (G * H)(a)$, where $(G * H)(a)$ is a group without any proper subgroups of finite index, and thus not F-A. □

We recall the definition of a Markov property.

**Definition 7.4.** We call an algebraic property of finitely presented groups $\rho$ a *Markov property* if there exist two finitely presented groups $G_+, G_-$ such that:
1. $G_+$ has the property $\rho$
2. $G_-$ does not have the property $\rho$, nor does it embed into any finitely presented group with the property $\rho$.

It is a result by Adian and Rabin (see [13, Theorem 3.3]) that no Markov property is algorithmically recognisable amongst finitely presented groups; this is usually the way one shows a given property is algorithmically unrecognisable. However, as the following corollary to theorem 7.3 shows, this technique cannot be used here.

**Corollary 7.5.** *Being F-A is neither a Markov property, nor a co-Markov property.*

We do not yet know if being F-A is an algorithmically recognisable group property amongst all finitely presented groups.

*Remark.* A set is said to be *recursively enumerable*, denoted r.e., if it is the domain of a total computable function; equivalently, the halting set of a partial-recursive function. It is an open problem from [2] as to whether having a non-trivial finite quotient is algorithmically recognisable amongst all finitely presented groups. If this could be done then the set of finite presentations of non-F-A groups would be r.e. as follows: Take a finite presentation $P = \langle X|R\rangle$. Enumerate the words $w_i \in X^*$. For each $i$, define the finite presentation $Q_i := \langle X|R, w_i\rangle$ and test if $\overline{Q}_i$ has any finite quotients. For each $i$, this procedure is recursive. Increasing $i$ by 1 each time, if we ever come across $Q_i$ where $\overline{Q}_i$ has no finite quotients then we halt, as $\overline{P}$ is not F-A. This process will halt if and only if $\overline{P}$ is not F-A.

## 8. Generalisation: n-Finitely Annihilated

8.1. **Motivation and definition.** It turns out we can generalise the definition of being F-A in a similar way to the definition of fully residually free groups from residually free groups (see [1]). Almost all of our results for F-A groups carry over to our new definition in some way.

**Definition 8.1** (c.f. definition 2.1)**.** Let $G$ be a group, and $n > 0$. A collection of $n$ elements $g_1, \dots, g_n \in G$ is said to be *finitely annihilated* if there is a non-trivial finite group $H$ and a surjective homomorphism $\phi : G \twoheadrightarrow H$ such that $\phi(g_i) = e$ for all $i = 1, \dots, n$. We say a non-trivial group $G$ is *$n$-finitely annihilated* ($n$-F-A) if every collection of $n$ elements in $G$ is finitely annihilated. From hereon, we insist that the trivial group is not $n$-F-A for any $n$.

Using the following definition of Brodie ([3]), we can come up with an equivalent interpretation of $n$-F-A groups, which turns out to be much more useful in the study of such groups.

**Definition 8.2.** An *$n$-covering* of a group $G$ is a collection of subgroups $\{N_i\}_{i\in I}$ over an index set $I$ such that, for any set of $n$ elements $\{g_1, \dots, g_n\} \subseteq G$, there is some $i \in I$ with $\{g_1, \dots, g_n\} \subseteq N_i$.

So an alternate equivalent interpretation of $n$-F-A is the following:

**Definition 8.3** (c.f. definition 2.2)**.** A group $G$ is $n$-F-A if and only if for each collection $g_1, \dots, g_n \in G$ there exists a proper, normal subgroup $N$ of finite index in $G$ such that $g_1, \dots, g_n \in N$. That is, $G$ has a proper $n$-covering by normal finite index subgroups (and hence maximal normal finite index subgroups).

It is then immediate that our definition of $n$-F-A groups really is a generalisation of F-A groups, in the following sense:

**Proposition 8.4.** *Let $G$ be $n$-F-A for some $n$. Then $G$ is $k$-F-A for every $k \leq n$.*

8.2. **Generalisation and extension of previous results.** We go over our results so far for F-A groups, and draw analogies to $n$-F-A groups. We state the most important of these, and provide proofs when it is not immediately obvious from the F-A case (where no proof is given, see the analogous case for F-A groups; the proof will be a straightforward adaptation).

**Proposition 8.5** (c.f. proposition 3.1)**.** *Let $G$ be a non-trivial group. Then $G$ is $n$-F-A if and only if neither of the following hold:*
1. $\mathrm{w}(G) \leq n$.
2. *There is some $g_1, \dots, g_n \in G$ such that $G/\langle\langle g_1, \dots, g_n \rangle\rangle^G$ has no proper finite index subgroups.*

Just as in the F-A case, being $n$-F-A is preserved under reverse quotients.

**Theorem 8.6** (c.f. theorem 4.1)**.** *Let $G$ be a group for which there is some quotient $G/H$ which is $n$-F-A. Then $G$ itself is $n$-F-A.*

Moreover, whenever we take a suitable quotient, being $n$-F-A is preserved.

**Theorem 8.7** (c.f. theorem 4.3)**.** *Let $G$ be a finitely generated $n$-F-A group, and $N \lhd G$. If $G = \bigcup_{i \in I} N_i$ is a proper $n$-covering by normal finite index subgroups, and $N$ is contained in every $N_i$, then $G/N$ is $n$-F-A.*

*Proof.* Take the quotient map $f : G \twoheadrightarrow G/N$. Then $f(N_i) = N_i/N$ will be normal, and finite index in $G/N$, as $f$ is a surjection. But since $N \lhd N_i$ by hypothesis, we have that $(G/N)/(N_i/N) \cong G/N_i$, hence $f(N_i)$ is also proper in $G/N$. So we have

$$G/N = f(G) = f\left(\bigcup_{i \in I} N_i\right) = \bigcup_{i \in I} f(N_i)$$

Moreover, if we take $g_1 N, \dots, g_n N \in G/N$, then there is some $j$ such that $g_1, \dots, g_n \in N_j$ and hence $g_1 N, \dots, g_n N \in N N_j/N = N_j/N = f(N_j)$. So $\bigcup_{i \in I} f(N_i)$ is our desired $n$-F-A covering of $G/N$. □

We now generalise the result by Brodie-Chamberlain-Kappe (theorem 5.1) to the case of $n$-coverings for finitely generated groups. This has been proved for non-finitely generated groups by Brodie in [3, Theorem 2.6]. We provide a simple proof here for the finitely generated case, and use it to prove the $n$-F-A analogue of our characterisation from theorem 5.13.

**Theorem 8.8.** *A finitely generated group $G$ has a finite proper $n$-covering $\bigcup_{i=1}^{k} N_i$ by normal finite index subgroups if and only if $\mathrm{w}(G^{\mathrm{ab}}) \geq n+1$ (equivalently, if and only if $G$ surjects onto an elementary $p$-group of rank $n+1$ for some prime $p$).*

*Proof.* We need only prove the forward direction (the reverse is implied by theorem 8.6). We proceed by induction, the case $n = 1$ is true by theorem 5.1. Now suppose $G = \bigcup_{i=1}^{k} N_i$ exhibits the $(n+1)$-F-A property. Then it also exhibits the $n$-F-A property, so $\mathrm{w}(G^{\mathrm{ab}}) \geq n+1$. Take $g \in G$ with $gG'$ in a generating set of minimal size for $G^{\mathrm{ab}}$. As $G$ is $(n+1)$-F-A, then for all $g_1, \dots, g_n$ there is an $N_j$ with $\{g, g_1, \dots, g_n\} \subseteq N_j$. So $G/\langle\langle g \rangle\rangle^G$ is $n$-F-A, and so has abelianisation of weight at least $n+1$. But then $G$ has abelianisation of weight at least $(n+1)+1 = n+2$ (as we annihilated $gG'$ which was in a minimal generating set for $G^{\mathrm{ab}}$), so the induction is complete. □

A straightforward generalisation of lemma 5.5 gives us the following lemma.

**Lemma 8.9** (c.f. lemma 5.5)**.** *A finitely generated abelian group $G$ has weight $n$ if any only if it surjects onto an elementary abelian $p$-group of weight $n$ for some prime $p$.*

And by combining the above two results, we deduce the following analogue of proposition 5.8.

**Proposition 8.10** (c.f. proposition 5.8)**.** *Let $G$ be a finitely generated abelian group. Then $G$ is $n$-F-A if and only if* $\mathrm{w}(G) \geq n+1$*.*

The rest of the results in this section can all be proved from their counterparts in section 5. We omit these proofs here, as they give no new insight to the theory of $n$-F-A groups not already obtained from F-A groups.

**Proposition 8.11** (c.f. proposition 5.3)**.** *Let $G$ be a finitely generated group with only finitely many distinct finite simple quotients. Then $G$ is $n$-F-A if and only if* $\mathrm{w}(G^{\mathrm{ab}}) \geq n+1$*.*

**Corollary 8.12** (c.f. corollary 5.4)**.** *A finite group $G$ is $n$-F-A if and only if* $\mathrm{w}(G^{\mathrm{ab}}) \geq n+1$*.*

**Proposition 8.13** (c.f. proposition 5.9)**.** *Let $G$ be a non-trivial finitely generated group whose finite simple quotients are all abelian (hence finite cyclic). Then $G$ is $n$-F-A if and only if* $\mathrm{w}(G^{\mathrm{ab}}) \geq n+1$*.*

Using the above results, we can provide the following characterisation for many $n$-F-A groups as follows:

**Theorem 8.14** (c.f. theorem 5.13)**.** *If $G$ is finitely generated and lies in at least one of the following classes, then $G$ is $n$-F-A if and only if* $\mathrm{w}(G^{\mathrm{ab}}) \geq n+1$*.*
1. *Simple*
2. *Free*
3. *Abelian*
4. *Solvable*
5. *Having only finitely many distinct finite simple quotients*
6. *Finite*

**Corollary 8.15** (c.f. corollary 7.5)**.** *Being $n$-F-A is neither a Markov property, nor a co-Markov property.*

## 9. Variations

In this section we begin a preliminary investigation of some variations on being F-A. An investigation by Brodie in [3, Section 3] is done on covering groups by subgroups with a specific property (i.e., abelian, nilpotent.) We instead look at coverings by normal subgroups, where the quotients by these groups have specific properties (i.e., simple, free, etc).

### 9.1. **Motivation.**

**Definition 9.1.** Given an algebraic property of groups $\rho$, we say that a group $G$ is *$\rho$-annihilated* ($\rho$-A) if, for each $e \neq g \in G$, there is a non-trivial group $H \in \rho$ and a surjective homomorphism $\phi_g : G \twoheadrightarrow H$ such that $\phi_g(g) = e$.

**Proposition 9.2.** *If $G$ has a quotient which is $\rho$-A, then $G$ itself is $\rho$-A.*

*Proof.* Let $p : G \twoheadrightarrow G/H$ be the projection map, and take some non-trivial $g \in G$. If $g \in H$, then take any non-trivial element $g'H \in G/H$ and non-trivial group $K \in \rho$ such that there is a surjective homomorphism $\phi_{g'H} : G/H \to K$ with $\phi_{g'H}(g'H) = e$. Then $\phi_{g'H} \circ p : G \to K$ is also surjective, and $\phi_{g'H} \circ p(g) = \phi_{g'H}(gH) = \phi_{g'H}(H) = e$ since $g \in H$. If $g \notin H$, then $gH \neq e$ in $G/H$. So again, we have a non-trivial $\rho$ group K and a surjective homomorphism $\phi_{gH} : G/H \to K$ with $\phi_{gH}(gH) = e$. Then $\phi_{gH} \circ p : G \to K$ is also surjective, and $\phi_{gH} \circ p(g) = \phi_{gH}(gH) = e$. Thus $G$ is $\rho$-A. ☐

### 9.2. **Free-Annihilated groups.**

We can begin to investigate free-A groups; it turns out they are relatively easy to characterise in the finitely generated case, which follows immediately from a result in [12].

**Theorem 9.3** ([12, p.57])**.** *Every finitely generated free group has a 1-relator quotient which has no free quotients itself. Hence, given a presentation $P = \langle x_1, \dots, x_n \mid - \rangle$ for $F_n$, there is a word $w_n \in \{x_1, \dots, x_n\}^*$ such that $\overline{\langle x_1, \dots, x_n \mid w_n \rangle}$ is not free.*

**Proposition 9.4.** *No finitely generated free group is free-A (and by 'free', we mean 'free and non-abelian').*

*Proof.* Assume $F_n$ is free-A for some fixed $n$; we proceed by contradiction. Take the word $w_n$ from theorem 9.3. Then $F_n/\langle\langle w_n \rangle\rangle^{F_n}$ has a non-abelian free quotient, contradicting theorem 9.3. ☐

**Corollary 9.5.** *Finitely generated free-A groups do not exist.*

*Proof.* Assume $G$ was free-A; we proceed by contradiction. Since $G$ is finitely generated, it can be written as the quotient of some finitely generated free group $F_n$. But this would imply that $F_n$ were itself free-A by proposition 9.2, contradicting the above result. ☐

So perhaps it would be more enlightening to allow 'free' to include $\mathbb{Z}$ (for then such groups definitely exist). But then it is straightforward to deduce the following observation:

**Proposition 9.6.** *A finitely generated group $G$ is free-A (where free includes $\mathbb{Z}$) if and only if $G^{\mathrm{ab}}$ has free rank $\geq 2$. That is, if and only if $G$ surjects onto $\mathbb{Z} \times \mathbb{Z}$.*

9.3. **Abelian-Annihilated and Simple-Annihilated groups.** One could also investigate abelian-A groups, simple-A groups, linear-A groups, and so on. Some of these are very easy to characterise, as we see here.

**Proposition 9.7.** *Let $G$ be a finitely generated group. Then $G$ is Abelian-A if and only if $G^{\text{ab}}$ is non-cyclic.*

*Proof.* It is straightforward to see that any finitely generated abelian group is abelian-A if and only if it is non-cyclic. So if $G^{\text{ab}}$ is non-cyclic, then it is abelian-A, hence so is $G$ by proposition 9.2. Conversely, suppose $G^{\text{ab}}$ is cyclic, and let $gG'$ be the generator of $G^{\text{ab}}$. If $G$ was abelian-A, then $G/\langle\langle g\rangle\rangle^G$ would have a non-trivial abelian quotient. But this is impossible as the abelianisation of $G/\langle\langle g\rangle\rangle^G$ is trivial. □

**Proposition 9.8.** *A group $G$ is Simple-A if and only if every $g \in G$ lies in a maximal normal subgroup.*

*Proof.* $G$ is Simple-A if and only if for every non-trivial $g \in G$ there is a non-trivial simple group $S_g$ and a surjection $\phi_g : G \twoheadrightarrow S_g$ annihilating $g$; this occurs if and only if for every non-trivial $g \in G$ there is a normal subgroup $N_g \triangleleft G$ containing $g$ such that $G/N_g$ is simple (equivalently, $N_g$ is maximal normal). □

## References


[1] B. Baumslag, *Residually free groups*, Proc. London Math. Soc. **17**, no. 3, 402–418, (1967).
[2] M. R. Bridson, *The Schur multiplier, profinite completions and decidability*, Bull. London Math. Soc. (3) **42**, 412–416 (2010).
[3] M. A. Brodie, *Finite n-coverings of groups*, Arch. Math **63**, 385–392, (1994).
[4] M. A. Brodie, R. F. Chamberlain, L.-C. Kappe, *Finite coverings by normal subgroups*, Proc. A.M.S. **104**, no. 3, 669–674, (1988).
[5] P. de la Harpe, *Topics in Geometric Group Theory*, The University of Chicago Press, (2000).
[6] J. Hempel, *Residual finiteness for 3-manifolds*, in 'Combinatorial group theory and topology (Alta, Utah, 1984)', 379–396, Annals of Math. Study, **111**, Princeton University Press, Princeton, NJ, 1987.
[7] J. Howie, *A proof of the Scott-Wiegold conjecture on free products of cyclic groups*, J. Pure Appl. Algebra **173**, no. 2, 167–176, (2002).
[8] T. Hungerford, *Algebra*, Springer, (1974).
[9] *The Kourovka notebook*, Edited by V. D. Mazurov and E. I. Khukhro, Russian Acad. Sci. Siber. Div., Inst. Math., Novosibirsk, (2006).
[10] A. A. Klyachko, *A funny property of a sphere and equations over groups*, Comm. Alg. **21**, no. 7, 2555–2575, (1993).
[11] P. Kutzko, *On groups of finite weight*, Proc. A.M.S. **55**, no. 2, 279–280, (1976).
[12] R. Lyndon, P. Schupp, *Combinatorial Group Theory*, Springer, (2001).
[13] C.F. Miller III, *Decision problems for groups - survey and reflections*, in 'Algorithms and classification in combinatorial group theory', G. Baumslag and C. F. Miller III, Editors, M.S.R.I Publ. **23**, Springer (1992), 1-59.
[14] A.Yu. Ol'shanskii *Geometry of Defining Relations in Groups*, Springer (1991).
[15] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv:math/0303109v1 (2003).
[16] J.P. Serre, *Trees*, Springer, (2003).

Maurice Chiodo
Department of Mathematics and Statistics
The University of Melbourne
Parkville, VIC, 3010
AUSTRALIA
*m.chiodo@pgrad.unimelb.edu.au*